Conclusions

A feedback control law for performing planar rendezvous and docking maneuvers is presented. The control law uses a feedback linearization approach and guarantees tracking along an arbitrary fixed docking direction with a desired approach speed. Although the control scheme is based on the linear CW equations, a simulation of the nonlinear system shows that the control law results in accurate rendezvous and docking conditions. The feedback control law is relatively simple and therefore may be a potential onboard guidance scheme for autonomous docking maneuvers.

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Uncertainty Modeling in Aerospace Flexible Structures

Dario H. Baldelli* and Ricardo S. Sánchez Peña†
Universidad de Buenos Aires,
1063 Capital Federal, Argentina

Introduction

THIS Note describes an alternative modeling scheme for the uncertainty present in aerospace flexible structures by a nonconservative characterization of the regions containing the perturbed eigenvalues of the dynamical system. In general, the term *flexible structure* is commonly used for linear systems with oscillatory properties characterized by a strong amplification of harmonic signals at the natural frequencies, and its transfer function poles are complex conjugate, typically with a small real part.[1] For the case of aerospace applications, these structures will have many resonant low frequencies with damping properties of approximately 0.5% critical, and often they appear in closely spaced clumps throughout the control system bandwidth. Frequently, the natural frequencies and damping factors (ω_i and ζ_i) of the structural modes that are included in the model are not known exactly.

Such inaccuracies or errors can be represented in two different ways. In the frequency domain three common ways of describing unstructured uncertainty are used. The difference between the nominal model and the real plant can be presented as additive or multiplicative uncertainty. A constraint is that these families of models must have the same number of right-half plane poles as the nominal one. Another description[2] is based on the uncertainty on the coprime factors. These descriptions do not have the previous limitation but may lead to conservative representations for lightly damped structures.[3] In the time domain uncertainty modeling can be described by the model parameters. In particular, the uncertainty can be expressed by variations in the A, B, C, or D matrices of the state-space representation of the model, i.e., $G(s) \overset{s}{=} (A, B, C, D)$.

The approach adopted here is motivated by that of Smith,[4] which describes the uncertain model through a linear fractional transformation (LFT), where the uncertainty present in the damping factor and natural frequency of each flexible mode is modeled through perturbations that affect the eigenvalues of the real modal dynamic matrix of the model. Hence, a nonconservative uncertainty description of these type of structures can be used to design robust controllers by standard methods such as H_∞ or μ-synthesis.[5] The object of this work is to point out that the real block diagonal perturbations (highly structured uncertainty) derived from the natural frequencies and damping factors can be described with no conservativeness as unstructured uncertainty. Therefore, optimal controllers may be obtained for this class of uncertain models without having to take into consideration the real nature of the uncertainty and its particular structure. To illustrate this uncertainty modeling technique, the results are applied to the synthesis of controllers for a flexible structure that is well-known in the literature.[6] In this case they were designed to increase the damping in the lightly damped low-frequency modes without affecting its neglected higher-frequency dynamics.

Uncertainty Modeling and Perturbed Eigenvalue Regions

Let $A \in \mathbb{R}^{2n \times 2n}$ be a modal matrix with n pairs of complex conjugate nominal eigenvalues $\lambda_{i\pm} = \alpha_i \pm j\beta_i$ and $W \in \mathbb{R}^{2n \times 2n}$ a diagonal weighting matrix. Now consider the uncertainty as perturbations to the nominal system eigenvalues represented by a particular LFT that replaces A by $A + W\Delta$, where $\Delta = \mathrm{diag}(\Delta_i)$, $\Delta_i \in \mathbb{R}^{2\times 2}$, $\bar{\sigma}(\Delta_i) \le 1$, $i = 1, \ldots, n$, and $\bar{\sigma}(\cdot)$ denotes the maximum singular value. For the specific LFT to be used, the A matrix must be in real modal form:

$$A = \mathrm{diag}(A_1, \ldots, A_n), \qquad A_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix} \tag{1}$$

$$W = \mathrm{diag}(w_1 I_2, \ldots, w_n I_2), \qquad w_i > 0 \tag{2}$$

The matrix W defines $2n$ disks in the complex plane centered at the eigenvalues of A, where w_i represents the disks radii located at $\lambda_{i\pm}$. Now, consider the problem of describing the uncertain model with inputs u and outputs y via an upper LFT formulation $F_u[Q(s), \Delta]$ so that all uncertainty can be represented as

$$y = \{C[sI - (A + W\Delta)]^{-1}B + D\}u \tag{3}$$

where

$$F_u[Q(s), \Delta] = Q_{21}\Delta(I - Q_{11}\Delta)^{-1}Q_{12} + Q_{22} \tag{4}$$

and the state-space realization of $Q(s)$ can be written as

$$Q(s) \overset{s}{=} \left[\begin{array}{c|cc} A & W & B \\ \hline I & 0 & 0 \\ C & 0 & D \end{array}\right] \tag{5}$$

As can be seen from Eq. (3), $F_u[Q(s), \Delta]$ clearly allows the inclusion of perturbations to the matrix A, mapping it to $A + W\Delta$. The contribution of this work is an extension of Theorem 4 from Ref. 4 to structured real block perturbations that appear in modal realizations of large space structures (LSSs); it provides a nonconservative description of the preceding uncertain model by means of unstructured uncertainty. This extension is implicit in the proof of the theorem mentioned before, although it is not explicitly stated. Next, we show this result.

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*Graduate Student, Dept. de Ingeniería Electrónica, Facultad de Ingeniería, Paseo Colón 850; currently Graduate Student, Facultad Regional Haedo, Dept. de Ingeniería Aeronautica, Universidad Tecnológica Nacional, 1706 Paris 532, Argentina. Member AIAA.

†Professor, Dept. de Ingeniería Electrónica, Facultad de Ingeniería, Paseo Colón 850; currently Professor, Comisión Nacional de Actividades Espaciales (CONAE), Paseo Colón 751, 1063 Capital Federal, Argentina.

Given A and W as prescribed by Eqs. (1) and (2), the structured and unstructured uncertainty described by Eqs. (6) and (7), respectively, are equivalent to the disk description in Eq. (8), as follows:

$$\{z \mid z \in \Lambda(A + W\Delta),\ \Delta \in \boldsymbol{\Delta}\} \tag{6}$$

$$= \{z \mid z \in \Lambda(A + W\Delta),\ \Delta \in \mathbb{C}^{2n\times 2n}, \bar{\sigma}(\Delta) \le 1\} \tag{7}$$

$$= \left\{ z \mid z \in \bigcup_{i=1}^{n} D_{i\pm} \right\} \tag{8}$$

where $\boldsymbol{\Delta}$, $D_{i\pm}$ is defined as follows:

$$\boldsymbol{\Delta} = \left\{ \Delta \middle| \Delta = \operatorname{diag}(\Delta_i),\ \Delta_i \in \mathbb{R}^{2\times 2}, \quad \Delta_i = \begin{bmatrix} \delta_{\alpha_i} & -\delta_{\beta_i} \\ \delta_{\beta_i} & \delta_{\alpha_i} \end{bmatrix}, \begin{matrix} \bar{\sigma}(\Delta_i) \le 1 \\ i = 1, \ldots, n \end{matrix} \right\} \tag{9}$$

$$D_{i\pm} = \{z : |z - \lambda_{i\pm}| \le w_i,\ i = 1, \ldots, n\} \tag{10}$$

and $\Lambda(\cdot)$ is the set of eigenvalues. Here $D_{i\pm}$ corresponds to the complex conjugate pair of disks in the complex plane of radius w_i and centered at $\lambda_{i\pm}$. Furthermore, if a set of k disks in Eq. (8) has no intersection with the remaining $(n - k)$ disks, then k eigenvalues are located within this set of k disks.

Clearly Eqs. (7) and (8) are equal to the result of Theorem 1 in Ref. 4. Because of the fact that $\Delta_i^T \Delta_i = (\delta_{\alpha_i}^2 + \delta_{\beta_i}^2) \times I_2$, then for $\operatorname{diag}(\Delta_i) \in \boldsymbol{\Delta}$ we have $\bar{\sigma}(\Delta_i) \le 1 \Leftrightarrow \delta_{\alpha_i}^2 + \delta_{\beta_i}^2 \le 1, i = 1, \ldots, n$. Therefore $\{W\Delta \mid \Delta \in \boldsymbol{\Delta}\}$ represents all closed circles of radius $w_i, i = 1, \ldots, n$. Considering that A_i and $w_i\ \Delta_i$ have the same structure, then $\{z \mid z \in \Lambda(A + W\Delta),\ \Delta \in \boldsymbol{\Delta}\}$ is the set of disks centered at $(\alpha_i, \pm\beta_i)$ of radius w_i, i.e., $D_{i\pm}$. This proves Eq. (8) $\Leftrightarrow$ Eq. (6), the latter being a more structured uncertainty description than the one mentioned in Theorem 4 from Ref. 4, especially useful for modal realizations of LSS.

The objective of this result is to describe the highly structured real block uncertainty represented by $\boldsymbol{\Delta}$ as unstructured complex uncertainty without any conservativeness. Note that the experimental measurements of the elastic modes produce an experimental frequency response from which we may obtain either the sets of real and imaginary parts or the (ω_i, ζ_i) pairs to represent the flexible structure dynamics. The choice is up to the designer's preference; there is no natural representation. Based on these arguments, we may adopt as the uncertainty set n circles covering the clusters of eigenvalues representing the elastic modes with no explicit conservativeness.

General Aerospace Flexible Structure Uncertain Model

In structural engineering a reasonable model for an aerospace flexible structure contains a large number of elastic modes and is usually obtained from finite dimensional approximations, such as the finite element method or through experimental identification. Both approaches provide reasonably accurate estimates of the frequencies and mode shapes only for the first few modes. As discussed in Ref. 4, only the low-frequency modes within the bandwidth of control can be modeled by the eigenvalue perturbation approach because according to the uncertainty radius an open-loop unstable model could be obtained. To avoid this situation, the center of the disks can be moved to locations with higher damping than the ones corresponding to the nominal eigenvalues.

The nominal model of the flexible structure can be divided in two parts. The first contains only the low-frequency modes, which via LFT can include any estimate or experimentally identified information about the range of variation of the structural parameters (ζ_i and ω_i) by means of the radii w_i of the disks $D_{i\pm}$. In the second the remainder elastic modes constitute the high-frequency uncertainty part of the uncertain model and will be covered by an additive weight $W_a(s)$. Usually, the latter presents a small relative magnitude at low frequencies, which reflects the fact that the nominal model represents the physical plant accurately in this part of the bandwidth. The final uncertainty model is

$$G_{\Delta}(s) = F_u[Q(s), \Delta_{\text{fm}}] + W_a(s)\Delta_a \tag{11}$$

with $\bar{\sigma}(\Delta_a) \le 1$, $\Delta_a \in \mathbb{C}^{r\times r}$ and $\bar{\sigma}(\Delta_{\text{fm}}) \le 1$, $\Delta_{\text{fm}} \in \mathbb{C}^{2n\times 2n}$. Here n is the number of elastic modes of the nominal model $G(s)$, and r is the dimension of the input and output vectors to Δ_a. The uncertainty represented by Δ_{fm} is selected as a complex dynamic but represents with no conservativeness the uncertain elastic modes. Finally, to increase the damping in the low-frequency elastic modes, a performance weight $W_p(s)$ is incorporated. It is used to minimize the resonance peaks of the first low-frequency modes, as compared with their open-loop response values.

Application Case

The method proposed in this work will be applied to the NASA Control–Structure Interaction (CSI) phase 0 model.[6] This example illustrates the use of our result in practical applications. It exhibits many of the characteristics of a typical aerospace flexible structure, i.e., many low-frequency and densely packed modes, structural interaction among components, and small structural damping. The structure consists of two vertical towers and two horizontal booms attached to the main section of 15.24 m. A reflector is mounted in the shorter vertical tower, and a laser source is located at the top of the other tower. Eight proportional bidirectional gas jets provide the control action, whereas eight collocated accelerometers give the output measurements. The nominal model of this structure is derived from a finite element formulation and has six modes because of its suspension device and 80 actual elastic modes. A reduced-order model consisting of 25 elastic modes is obtained from a controllability/observability elastic modes analysis. For the CSI model the analytical natural frequencies are accurate within a 0.1%, whereas the nominal damping factors in the order of 0.5% below 2 Hz are precise within 1%. Therefore, the regions generated from a 0.1% error in the natural frequencies and 1% in the damping factors for the first nine lowest frequency modes are modeled via eigenvalue perturbation techniques and covered by a constant diagonal matrix $W \in \mathbb{R}^{18\times 18}$, i.e.,

$$W = \operatorname{diag}\{\omega_i \times I_2\}$$

$$[\omega_1 \quad \cdots \quad \omega_9]$$

$$= [9 \quad 9 \quad 10 \quad 46 \quad 47 \quad 55 \quad 93 \quad 109 \quad 118] \times 10^{-4}$$

The remaining 16 structural modes are considered as additive uncertainty and are covered by the weight $W_a(s)$:

$$W_a(s) = 0.11 \left[\frac{(s + 1.5)32}{(s + 32)1.5} \right]^3 I_{8\times 8} \tag{12}$$

The performance transfer matrix $W_p(s)$ is chosen to penalize the resonance peaks of the first nine lower-frequency modes and is as follows:

$$W_p(s) = \frac{1}{130} \left[\frac{10}{(s + 10)} \right] I_{8\times 8} \tag{13}$$

Here, we compute two different controllers by μ-synthesis. The reader is referred to Ref. 5 for further details on this procedure. For the controller $K_1(s)$ the uncertainties related with the lower elastic modes are modeled using a full block $\Delta_{f1} \in \mathbb{C}^{18\times 18}$, whereas in $K_2(s)$ nine perturbations of rank 2 are used, i.e., $\Delta_{f2} = \operatorname{diag}(\Delta_i)$, $\Delta_i \in \mathbb{C}^{2\times 2}$ with $\bar{\sigma}(\Delta_i) \le 1$, $i = 1, \ldots, 9$. We begin with an initial H_∞ design, which reaches a μ value of 461.46, far from the desired unity value. After one $D-K$ iteration and through first-order D scalings, the controllers were synthesized. Both were reduced to an order of 73 by balanced stochastic truncation[7]; this was accomplished with a degradation effect less than 0.01% on the closed-loop μ value. The closed-loop transfer matrices for $K_1(s)$ and $K_2(s)$ achieve μ values of nearly 1, as shown in Fig. 1. These robust performance plots for Δ_{f1} and Δ_{f2} present an equivalent behavior, and the maximum value is obtained at the same frequency. Also, nominal performance results using the 25 degree-of-freedom (DOF) model (not shown) indicate a similar reduction of the peaks heights for the first nine elastic modes achieved with both controllers, while the remaining 16 modes are left unchanged because of the controller gain roll-off across them.

Worst-case performance degradation plots, which show the trade-offs between size of uncertainty and worst-case performance, are presented in Fig. 2. Worst-case perturbations with block structures

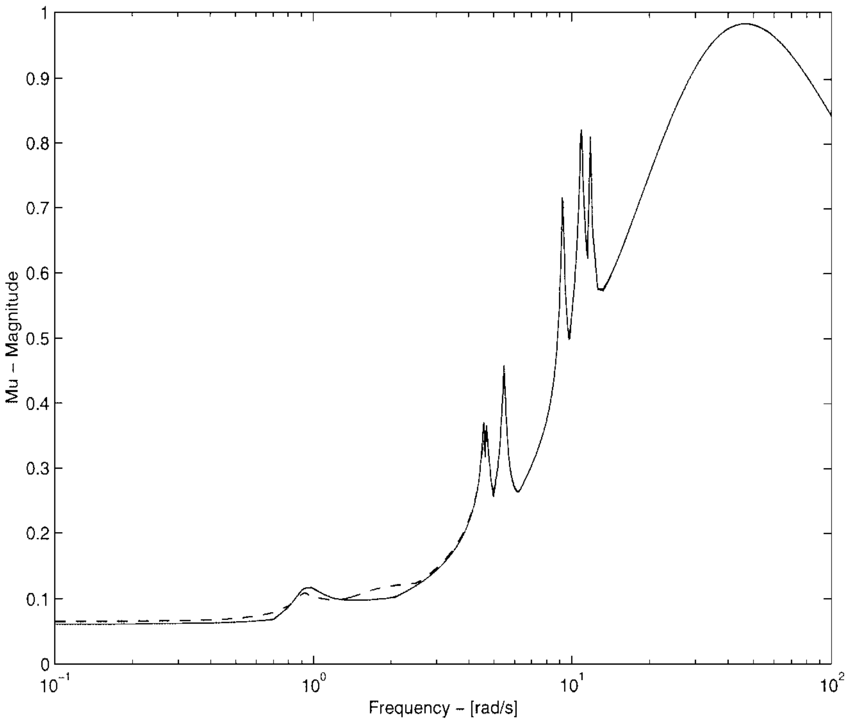

Fig. 1 Robust performance plots: - - -, Δ_{f1}, and ——, Δ_{f2}.

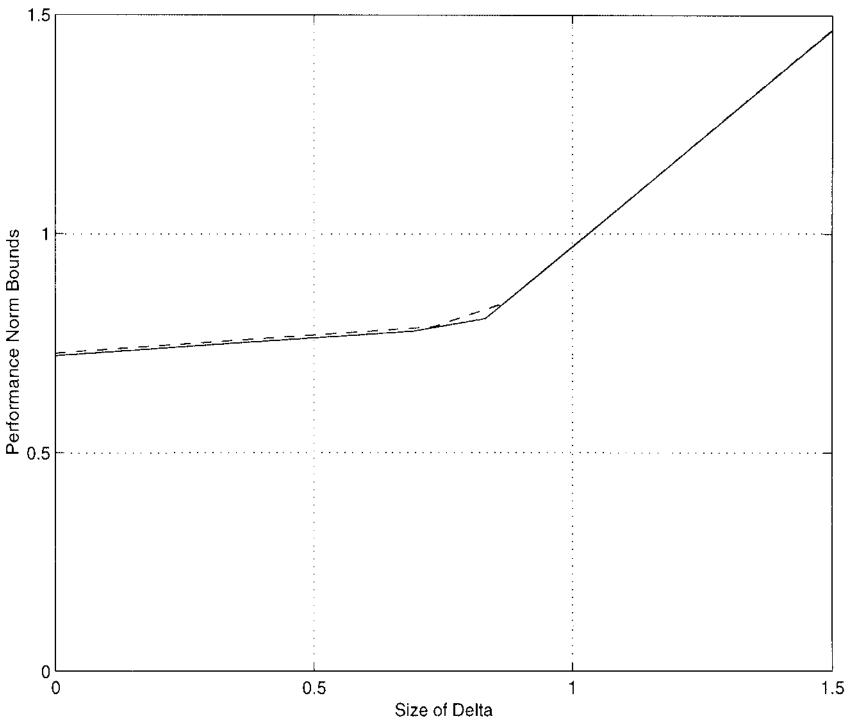

Fig. 2 Worst-case performance degradation: - - -, $K_1(s)$, and ——, $K_2(s)$.

as Δ_{f1} and Δ_{f2} of several sizes were generated[5] using the nine-DOF model in $G_\Delta(s)$, closed-loop interconnected with $K_1(s)$ and $K_2(s)$. These plots show that a near identical robust performance condition is achieved for the uncertain plants $G_\Delta(s)$ generated with a complex dynamic perturbation (Δ_{f1}) and those generated with the more stringent structured representation (Δ_{f2}). Note that according to our result an even more structured description could have been adopted for the elastic modes, i.e., $\Delta \in \mathbf{\Delta}$ as defined in Eq. (9). We should note that the exact optimal synthesis procedure for general structured real uncertainty is a problem that cannot be solved by polynomial time algorithms (NP-hard problem).

Conclusions

This work has presented an alternative description of the uncertainty associated with the natural frequency and damping factor of each elastic mode in an aerospace flexible structure. It has been proved that an unstructured perturbation of full rank and a highly structured real block diagonal perturbation, affecting the nominal model, produce an equivalent description of the set of eigenvalues associated with the elastic modes. This represents a very useful result for the design of controllers by the H_∞/μ-synthesis methodology. If only robust stability needs to be achieved, an H_∞ optimal controller solves the problem because the elastic modes are described as global dynamic uncertainty. In the case where robust performance is sought, we may even incorporate an additive uncertainty description for the unknown higher-frequency modes, and only three uncertainty blocks will be needed, which allows an exact computation of the μ seminorm.

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Weighted-Residual Discretization for Uniform Damping and Uniform Stiffening of Structural Systems

Larry Silverberg*
North Carolina State University, Raleigh, North Carolina 27695-7910
and
Gregory Washington†
Ohio State University, Cleveland, Ohio 43210-1107

Introduction

LINEAR feedback control methods for full-dimensional control of structural systems are well established. One accepted approach is to control the structure's motion by controlling its modes.[1] The designer first prescribes a desirable dynamic performance for each controlled mode and then synthesizes a full-dimensional control from the modal control forces. Another approach, which is particularly robust in the presence of modeling error and actuator failure, is called decentralized control or local control.[2] Using this approach, the full-dimensional control forces and the sensor measurements are related by a decentralized (local) control algorithm. Still another approach is to employ linear optimal control theory.[3] Each approach has merit—the first approach placing highest priority on dynamic performance, the second approach placing highest priority on design simplicity and robustness, and the third approach placing highest priority on optimality.

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*Professor, Mechanical and Aerospace Engineering. Member AIAA.
†Assistant Professor, Mechanical Engineering.

In an attempt to satisfy the requirements of all three approaches, it was later shown that uniform damping of the structure's natural modes of vibration leads to a local control that is near globally optimal.[4] The limitation of uniform damping is that it is only capable of controlling settling time and not capable of controlling peak-overshoot and steady-state error. This Note extends the uniform damping results given in Ref. 4, first showing that uniform damping and uniform stiffening of the structure's natural modes of vibration leads to a local control. The three settling-time, peak-overshoot, and steady-state error requirements are satisfied as well.

The development of the uniform damping and uniform stiffening control algorithm proceeds by first considering distributed control forces. The distributed control forces are then discretized in order to realize the uniform damping and uniform stiffening by means of discrete forces. The method of discretizing the controls that is developed in this Note is a weighted residual method. The method is capable of turning local distributed control forces into either local discrete control forces or into global discrete control forces, depending on the admissible functions used in the discretization. A numerical example shows the discretization of local distributed forces into global discrete forces.

Modal Control

The vibration of a normal-mode structural system is governed by the linear differential equation

$$\rho(\boldsymbol{P})\frac{\mathrm{d}^2\boldsymbol{u}(\boldsymbol{P},t)}{\mathrm{d}t^2}+\boldsymbol{L}\boldsymbol{u}(\boldsymbol{P},t)=\boldsymbol{f}_C(\boldsymbol{P},t)+\boldsymbol{f}_D(\boldsymbol{P},t) \tag{1}$$

where $\rho(\boldsymbol{P})$ denotes mass density at point $\boldsymbol{P}$ in the domain D of the structural system, $\boldsymbol{u}(\boldsymbol{P},t)$ denotes displacement at point P and time t, $\boldsymbol{L}$ is a self-adjoint linear operator expressing structural stiffness, $\boldsymbol{f}_C(\boldsymbol{P},t)$ is a control force, and $\boldsymbol{f}_D(\boldsymbol{P},t)$ is a quasistatic external disturbance.[5] The linear feedback control force has the general form

$$\boldsymbol{f}_C(\boldsymbol{P},t)=-\boldsymbol{G}\boldsymbol{u}(\boldsymbol{P},t)-\boldsymbol{H}\frac{\mathrm{d}\boldsymbol{u}(\boldsymbol{P},t)}{\mathrm{d}t}-\boldsymbol{I}\int\boldsymbol{u}(\boldsymbol{P},t)\,\mathrm{d}t \tag{2}$$

where $\boldsymbol{G}$, $\boldsymbol{H}$, and $\boldsymbol{I}$ denote proportional feedback, derivative feedback, and integral feedback linear operators, respectively. The structural system exhibits normal-mode behavior. Accordingly, the displacement $\boldsymbol{u}(\boldsymbol{P},t)$ is expressed as an infinite sum of natural modes of vibration $\phi_s(\boldsymbol{P})$ multiplied by modal displacements $\eta_s(t)$, as

$$\boldsymbol{u}(\boldsymbol{P},t)=\sum_{s=1}^{\infty}\phi_s(\boldsymbol{P})\eta_s(t)$$

Substituting this into Eq. (1), multiplying the result by

$$\int_D\phi_r(\boldsymbol{P})\cdot(\,)\,\mathrm{d}D$$

and invoking the orthonormality conditions

$$\int_D\rho(\boldsymbol{P})\phi_r(\boldsymbol{P})\cdot\phi_s(\boldsymbol{P})\,\mathrm{d}D=\delta_{rs}$$

$$\int_D\boldsymbol{L}\phi_r(\boldsymbol{P})\cdot\phi_s(\boldsymbol{P})\,\mathrm{d}D=\omega_r^2\delta_{rs},\qquad(r,s=1,2,\ldots)$$

we get the modal equations of motion

$$\frac{\mathrm{d}^2\eta_r(t)}{\mathrm{d}t^2}+\eta_r(t)=N_{\mathrm{Cr}}(t)+N_{\mathrm{Dr}} \tag{3a}$$

where

$$N_{\mathrm{Cr}}(t)=\int_D\phi_r(\boldsymbol{P})\cdot\boldsymbol{f}_C(\boldsymbol{P},t)\,\mathrm{d}D \tag{3b}$$

$$N_{\mathrm{Dr}}=\int_D\phi_r(\boldsymbol{P})\cdot\boldsymbol{f}_D(\boldsymbol{P},t)\,\mathrm{d}D \tag{3c}$$

denote modal control forces and modal disturbance forces, respectively. Next, substitute Eq. (2) into Eq. (3b) to yield the modal control algorithm

$$N_{\mathrm{Cr}}(t)=-\sum_{s=1}^{\infty}\left[g_{rs}\eta_s(t)+h_{rs}\frac{\mathrm{d}\eta_s(t)}{\mathrm{d}t}+i_{rs}\int\eta_s(t)\,\mathrm{d}t\right] \tag{4}$$